A front–tracking alternative to the random choice method

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Abstract. An alternative to Glimm's proof of the existence of solutions to systems of hyperbolic conservation laws is presented. The proof is based on an idea by Dafermos for the single conservation law and in some respects simplifies Glimms original argument. The proof is based on construction of approximate solutions of which a subsequence converges. It is shown that the constructed solution satisfies Lax' entropy inequalities. The construction also gives a numerical method for solving such systems

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1. Introduction. We consider the initial value problem for the general system of hyperbolic conservation laws

$$u_t + f(u)_x = 0.$$

Our analysis is based on Lax' [**1**] solution of the Riemann problem. We here give an alternative proof of Glimm's fundamental result [**2**] not based on a random sequence. Since Glimm's paper there has been few generalizations of his result, but Liu [**3**] showed that Glimm's proof did not actually depend on the random sequence, and that it converged for any equidistributed sequence. Chorin [**4**] developed Glimm's construction into a numerical method. Using Glimm's construction, Lax [**5**] showed that the conctructed solution satisfied the entropy inequalities provided the system admitted an additional conservation law. This system of equations models a diverse range of physical phenomena, e.g., traffic flow [**6**], gas dynamics [**7**] and multi-phase flow in porous media [**8**].

Our proof is based on ideas from the study of the single conservation law. Dafermos [**9**] used a piecewise linear continous approximation to the flux function f to obtain approximate solutions containing only shocks. This idea was further developed into a numerical method by LeVeque [**10**] and by Holden *et al.* [**11**], and was generalized to several space dimensions by Høegh–Krohn and Risebro [**12**].

We construct our solutions by starting with an approximation to the solution of the Riemann problem where the rarefaction part of the solution is replaced by an approximating step function. The inital value function is also approximated by a step function which gives a series of Riemann problems. Each discontinuity in the approximate solution is then tracked until it interacts with other discontinuities. For such interactions we can use some of the estimates in [**2**] directly, and we here only give the differences from Glimm's proof. Our main result is that if the total variation of the initial data is small, then a weak solution of the initial value problem exists. Without the assuming the existence of an additional conservation law, it is easy to show that our constructed solution satisfies Lax' entropy inequalities, and therefore is not of what Glimm [**2**] called "extranous" type. The construction in a natural way defines a numerical method for solving hyperbolic conservation laws. For general background we refer the reader to [**13**, part 3], and the references quoted there.

2. Method and notation. We will consider the equation

$$\begin{aligned} u_t + f(u)_x &= 0 \\ u_0(x) &= u(x,0) \end{aligned} \tag{2-1}$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is strictly hyperbolic, that is, the Jacobian df has real eigenvalues $\lambda_1(u), \dots, \lambda_n(u)$ such that

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u).$$

$u \in \mathbf{R}^n$ is to be the *weak solution* of (1-1):

$$\int_0^\infty \int_{-\infty}^\infty \Big(\phi_t u + \phi_x f(u)\Big) dx dt + \int_{-\infty}^\infty \phi(x,0) u_0(x) dx = 0 \tag{2-2}$$

for all smooth ϕ with compact support in (x,t).

We define the Riemann problem for (2-1) to be the initial value problem where

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases} \tag{2-3}$$

The solution of this Riemann problem consists of three ingredients; ***shocks, rarefaction waves and contact discontinuities***. For an explaination of these see [**13**, chapter 17].

THEOREM 1. *Let $u_l \in N \subset \mathbf{R}^n$ and suppose (2-2) is hyperbolic and that each characteristic field is either genuinly non-linear or linearly degenerate in N. Then there is a neighbourhood $M \subset N$ such that for $u_r \in M$, (2-3) has a solution. This solution consists of at most $n+1$ constant states separated by shocks, rarefaction waves and contact discontinuities. There is only one such solution in M.*

PROOF: See [**13**, theorem 17.18]. ∎

We will construct an approximation to this solution. Through each point in M we have n one parameter families of curves $U_k(u,\epsilon)$, $k = 1,\dots,n$. These have continuous derivatives of order two at $\epsilon = 0$, and have the property that if u_r is on $U_k(u_l,\epsilon)$ and the kth. field is genuinely nonlinear, u_r can be connected to u_l by a rarefaction wave iff $\epsilon > 0$ and by a shock iff $\epsilon < 0$, $U_k(u_l,0) = u_l$. We call $|\epsilon|$ the ***strength*** of the wave. If the kth. field is linearly degenerate $U_k(u,\epsilon)$ consists of the states that can be connected to u by a contact discontinuity.

We can therefore draw the solution in phase space M by drawing the $n+1$ constant states and the U_k curves between them. See figures 1 and 2. For a more detailed description of these concepts we again refer to [**13**].

The approximation we will make is the following: We start with the correct solution to (2-3). Leave each shock or contact discontinuity as it is. Along the rarefaction curves, we fix an initial $\delta > 0$, approximate the rarefaction fan by constant states $u_i^{(k)} = U_k(u_{i-1}^{(k)}, i\delta)$ for $i = 1,\dots,m$, where m is chosen such that $u_{m+2}^{(k)}$ is "past" the next constant state in the solution: u^{k+1}. $u_i^{(k)}$ and $u_{i+1}^{(k)}$ will be separated by a discontinuity moving with speed $\lambda_k(u_{i+1}^{(k)})$. This approximation corresponds to making a step function approximation of $u(x,t)$ at each fixed t. We call our approximation $u_\delta(x,t)$. We have that

$$\lim_{\delta \to 0} u_\delta = u \text{ for all } t.$$

The limit is in $L_1(\mathbf{R}, dx)$ for each t. We have that u will satisfy (2-2), and since supp ϕ is confined to $t < T < \infty$, we have that

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t u_\delta + \phi_x f(u_\delta))\,dx\,dt + \int_{-\infty}^\infty \phi(x,0)u_0(x)\,dx \to 0$$

as $\delta \to 0$, since, by the bounded convergence theorem, $f(u_\delta) \to f(u)$ in L_1. Therefore u_δ is an approximate solution to (2-3).

Assume now that u_0 is a step function with a finite number of steps and compact support. We want to use our approximate solutions u_δ to generate a "solution" to the initial value problem with u_0 as the initial function. At each initial discontinuity we construct the approximating u_δ, when two of these interact at some $t > 0$, we are still in the class of step functions with compact support and a finite number of steps. Therefore the process can be repeated. With a slight abuse of notation we will call this "solution" u_δ.

It may be that one cannot continue this process to all finite times. This is the case if the collision times accumulate at some time t, but since the propagation speed of each discontinuity is finite, this must also be a local phenomenon in x. Thus the corresponding collision positions must converge. See figure 3.

If we therefore estimate the point to which the collisions converge, and stop tracking the discontinuities involved with this accumulation after a number of collisions, solve the Riemann problem with values immediatly to the left and right of the accumulation area, we will make an arbitrarely small error in L_1. Theoretically, however, we may assume that we can continue tracking discontinuities past a countable number of collisions.

3. Results. We follow the notation in [**13**,p. 370]. By

$$(u_l, u_r) = [(u_0, \dots, u_n)/(\varepsilon_1, \dots \varepsilon_n)] \tag{3-1}$$

we wean that u_k is connected to u_{k-1} by a k–shock or a k–rarefaction wave with strength $|\varepsilon_k|$, i.e. $u_k = U^{(k)}(u_{k-1}, \varepsilon_k)$. Let now u_l, u_m, u_r be given states near $\bar{u}$, and let

$$(u_l, u_m) = [(u_0, \dots, u_n)/(\alpha_1, \dots \alpha_n)] \tag{3-2}$$

$$(u_m, u_r) = [(u_0, \dots, u_n)/(\beta_1, \dots \beta_n)]. \tag{3-3}$$

With these definitions in hand we can prove the following slight modificantion of [**13**, theorem 19.2] or [**2**, theorem 2.1].

LEMMA 1. *Assume that a discontinuity α (in our scheme) separating (u_l, u_m) and a discontinuity β separating (u_m, u_r) collide, and that (2-1), (2-2) and (2-3) hold. Then*

$$\varepsilon_i = \alpha + \beta + O(1)|\alpha||\beta|$$

PROOF: The proof of this is the same as the proof of the theorem in [**2**]. ∎

We now define

$$T(t) = \sum |\varepsilon_i^j|$$

where the sum is taken over all discontinuities of u_δ at time t. We have that $T(t)$ is equivalent to the total variation of u_δ. We say that two discontinuities ***approach*** each other if they are neighbouring discontinuities and the speed of the one on the left is larger than the speed of the one on the right. Let

$$Q(t) = \sum |\alpha||\beta|$$

where the sum is taken over all approaching discontinuities α and β. Note that T and Q only change values when we have a collision.

THEOREM 2. *Let $t_1, t_2 > 0$. If $t_2 > t_1$ and $T(t_1)$ is sufficiently small, then*

$$Q(t_2) \leq Q(t_1)$$
$$T(t_1) + kQ(t_1) \leq T(t_2) + kQ(t_2)$$

for some $k > 0$.

PROOF: The proof is similar in spirit to the proof of the corresponding theorem for Glimm's construction, see **[2]** or **[13]** .

We first assume that $t_1 < t_2$ are such that there is only one collision time for u_δ in between t_1 and t_2. Let I, J be the intervals indicated in figure 4. ($\mathbf{R} = J \cup I$)

Let $T(t, I)$ be $T(t)$ with the summation restricted to I, similarely for $Q(t, I)$. From the lemma we have

$$T(t_2) \leq T(t_1) + k_0 Q(t_1, j)$$
$$Qt_2 = Q(t_2, I) + Q(t_2, I, J)$$

where $Q(t, I, J)$ is the sum with one wave from I and the other from J.

$$\begin{aligned} Q(t_2, I, J) &= \sum |\varepsilon_i||\delta| \\ &= \sum_{\delta \text{ appr. } \alpha \text{ or } \beta} (|\alpha| + |\beta|)|\delta| + k_0 Q(t_1, J) T(t_1, J) \\ &\leq Q(t_1, I, J) + k_0 Q(t_1, J) T(t_1) \\ &\leq Q(t_1, I, J) + 1/2 Q(t_1, J) \text{ if } k_0 T(t_1) \leq 1/2. \end{aligned}$$

Therefore

$$\begin{aligned} Q(t_2) - Q(t_1) &= [Q(t_2, I) + Q(t_2, I, J)] - [Q(t_1, I) + q(t_1, J) + Q(t_1, I, J)] \\ &\leq Q(t_1, I, J) + 1/2 Q(t_1, J) - Q(t_1) - Q(t_1, I, J) \\ &\leq -1/2 Q(t_1, J) \leq 0. \end{aligned}$$

Now

$$\begin{aligned} T(t_2) + kQ(t_1) &\leq T(t_1) + k_0 Q(t_1, J) + Q(t_1) - k/2 Q(t_2, J) \\ &\leq T(t_1) + kQ(t_1) \text{ if } k_0 - k/2 \leq 0. \end{aligned}$$

Summing we have that the inequalities hold for any $t_2 > t_1$. ∎

COROLLARY 1. *If $T.V.(u_0)$ is sufficiently small then*

$$\operatorname{osc} u_0 \leq T.V.(u_\delta) \leq cT(t) \leq \bar{c}T(o) \leq T.V.(u_0)$$

where all constants are independent of t and δ.

PROOF: osc $\leq T.V.$ is always true. $T.V. \leq cT$ since they are equivalent norms. $T(t) \leq T(t) + kQ(t) \leq T(0) + kQ(0) \leq T(0) + kT^2(0) \leq 2T(0)$ if $kT(0) \leq 1$. ∎

COROLLARY 2. *If $T.V.(u_0)$ is small then*

$$T.V._x(u_\delta) + \sup_x |u_\delta| \le cT.V.u_0$$

where c is independent of t and δ.

COROLLARY 3.

$$\|u_\delta(\cdot, t_1) - u_\delta(\cdot, t_2)\| \le c|t_2 - t_1|$$

where c is independent of δ, t_1 and t_2.

These two corollaries are consequences of corollary 1, and their proof may be found in [**13**,p.384].

Now we have that the u_δ functions satisfy

$$\|u_\delta(\cdot,\cdot)\|_\infty \le M_1 \tag{3-4}$$

$$T.V._x(u_\delta(\cdot, t)) \le M_2 \tag{3-5}$$

$$\|u_\delta(\cdot, t_1) - u_\delta(\cdot, t_2)\| \le M_3|t_2 - t_1|. \tag{3-6}$$

The constants M_i are independent of the δ and the times t_1 and t_2. Using Helly's theorem as in [**13**] one can show that (3-4) to (3-6) imply the following:

THEOREM 3. *If 2-4 to 2-6 hold then a subsequence converges in L_1^{loc}. For this subsequence $f(u_\delta) \to f(u)$ in L_1^{loc}, where u is the limit function.*

We want to use this theorem for the functions $\{u_\delta\}$ with the initial function u_0 in L_1. u_0 can be approximated by a step function with compact support and a finite number of steps; $u_{0,\Delta x}$. Let $u(x,t)$ be a limit as $\Delta x \to 0$ and $\delta \to 0$. For all suitable ϕ we define

$$\mathcal{I}_\phi(u, f) = \int_0^\infty \int_{-\infty}^\infty \Big(\phi_t u + \phi_x f(u)\Big) dx dt + \int_{-\infty}^\infty \phi(x, 0) u_0(x) dx$$

and

$$\mathcal{I}_\phi^{t_1,t_2}(u, f) = \int_{t_1}^{t_2} \int_{-\infty}^\infty \Big(\phi_t u + \phi_x f(u)\Big) dx dt$$
$$+ \int_{-\infty}^\infty \phi(x, t_1) u(t_1, x) dx - \int_{-\infty}^\infty \phi(x, t_2) u(t_2, x) dx$$

Now let t_1, t_2 be consecutive times when discontinuites of u_δ collide, let $v(x,t)$ be the weak solution of

$$v_t + f(v)_x = 0$$
$$v(x, t_1) = u_\delta(x, t_1).$$

We now fix δ and ϕ.

LEMMA 3.1. *For $t \in (t_1, t_2)$ we have that*

$$\int_{x_1}^{x_2} |v - u_\delta| dx = O(\delta)(t - t_1). \tag{3-7}$$

PROOF: Note that $v = u_\delta$ exept when (x, t) is in a rarefaction fan. In a rarefaction fan the difference $|u_\delta - v|$ is always less than or equal to δ. If $\hat{u}_r$, $\hat{u}_l$ are the states to the right and left of such a fan respectivly, then the integral across the fan will be a sum of integrals across each step of u_δ. See figures 5 to 7. The number of such steps is $\frac{|\hat{u}_r - \hat{u}_l|}{O(\delta)}$, and the width of each region where u_δ differ from v is $(t - t_1)O(\Delta\lambda)$, where $\Delta\lambda = | \lambda(u_i^{(k)}) - \lambda(u_{i+1}^{(k)}) | = O(\delta)$ Therefore (3-7) is a sum over all rarefaction fans of u_δ

$$\sum_i \frac{|\hat{u_{r_i}} - \hat{u_{l_i}}|}{O(\delta)} O(\delta)(t - t_1)O(\delta).$$

But this is less than

$$T.V.(u_\delta(x, t_1))(t - t_1)O(\delta)$$

and the result now follows from corollary 1. ∎

LEMMA 3.2. *If we let t_1, t_2 be as before we have*

$$\mathcal{I}_\phi^{t_1,t_2}(u_\delta, f) = O(\delta)((t_2 - t_1) + (t_2 - t_1)^2).$$

PROOF: Let $M \geq \sup\{|\phi_x|, |\phi_t|, |\phi|, |df|\}$, and let $v(x, t)$ be as before, then

$$\begin{aligned}
\mathcal{I}_\phi^{t_1,t_2}(u_\delta. f) &= \mathcal{I}_\phi^{t_1,t_2}(u_\delta, f) + \mathcal{I}_\phi^{t_1,t_2}(v, f) \\
&= \int_{t_1}^{t_2} \int ((u_\delta - v)\phi_t + (f(u_\delta) - f(v)\phi_x) dx dt - \int \phi(x, t_2)(u_\delta - v) dx \\
&\leq M\Big(\int_{t_1}^{t_2} \int |u_\delta - v| dx dt + \int |u_\delta - v| dx\Big) + \int_{t_1}^{t_2} \int \big|f(u_\delta) - f(v)\big| dx dt. \qquad (3\text{-}9)
\end{aligned}$$

Now

$$f(u_\delta) - f(v) = df(u_\delta - v) + O^2(u_\delta - v)$$

and

$$\int \big|f(u_\delta) - f(v)\big| dx \leq M \int |u_\delta - v| dx + \int \big|O^2(u_\delta - v)\big| dx \tag{3-10}$$

Using lemma 3.1 on (3-9) and (3-10) and integrating (3-8) will give lemma 3.2. ∎

LEMMA 3.3.

$$\lim_{\delta \to 0} \mathcal{I}_\phi(u_\delta, f) = 0.$$

PROOF: If we let t_i, t_{i+1} be consecutive times when discontinuities collide we have

$$\mathcal{I}_\phi(u_\delta, f) = \sum_i \mathcal{I}_\phi^{t_i, t_{i+1}}(u_\delta, f),$$

therfore by lemma 3.2

$$\mathcal{I}_\phi(u_\delta, f) = O(\delta) \sum ((t_{i+1} - t_i) + (t_{i+1} - t_i)^2) \tag{3-12}$$

where $\sum(t_{i+1} - t_i) \leq T$ and T is such that supp ϕ is contained in $\{t \leq T\}$. Therefore the sum in (3-12) is finite and the lemma follows. ∎

Thus u_δ converges to a weak solution.

THEOREM 4. *Assume $f : \mathbf{R}^n \to \mathbf{R}^n$ is strictly hyperbolic, and $u_0 : \mathbf{R} \to \mathbf{R}^n$ is such that $T.V._x(u_0)$ is sufficiently small. Then there exists a weak solution $u(x,t)$ to the initial value problem*

$$\begin{aligned} u_t + f(u)_x &= 0 \\ u_0(x) &= u(x, 0). \end{aligned}$$

For the solution of the Riemann problem we have that all discontinuities satisfy the Lax entropy conditions:

$$\begin{aligned} \lambda_k(u_r) < s_k \leq \lambda_{k+1}(u_r) \\ \lambda_{k-1}(u_l) \leq s_k < \lambda_k(u_l) \end{aligned} \tag{3-13}$$

where s_k is the speed of the k-shock. For our approximation u_δ, we have that (3-13) is satisfied for the shocks, i.e. at least for all discontinuities of magnitude greater than δ. Therefore a limit function has to satisfy (3-13) for discontinuities of any magnitude. We therefore have the following corollary.

COROLLORY 4. *The discontinuities of $u = \lim_{\delta \to 0} u_\delta$ satisfy the Lax entropy conditions (3-13).*

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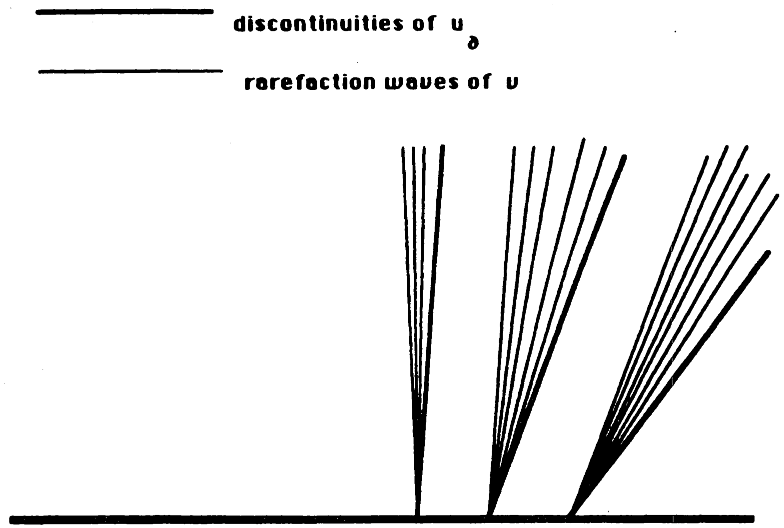

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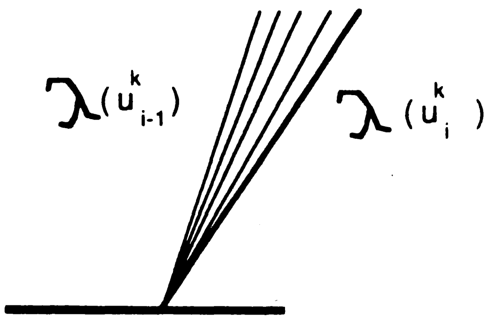

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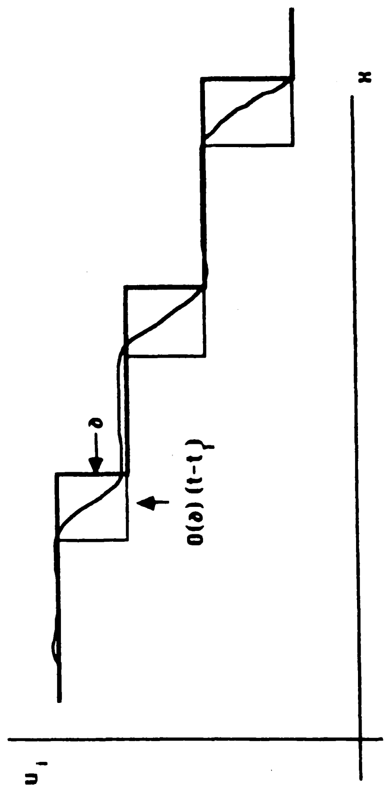

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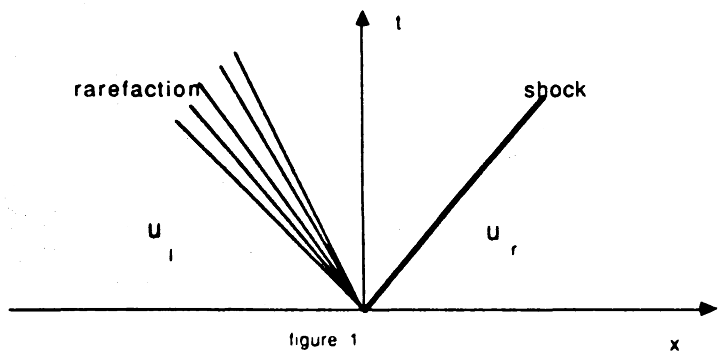


figure 1

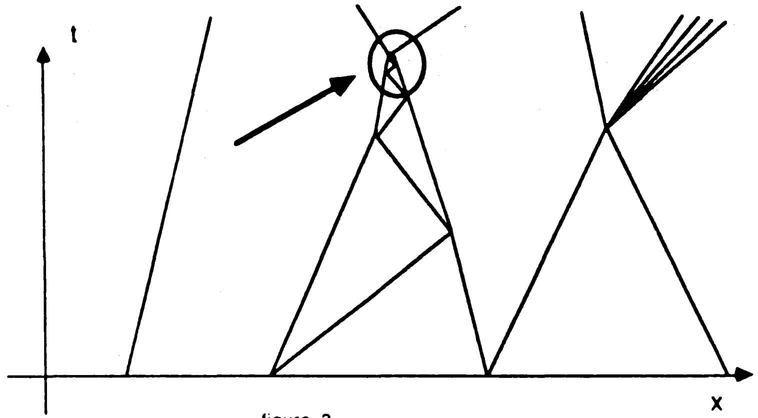
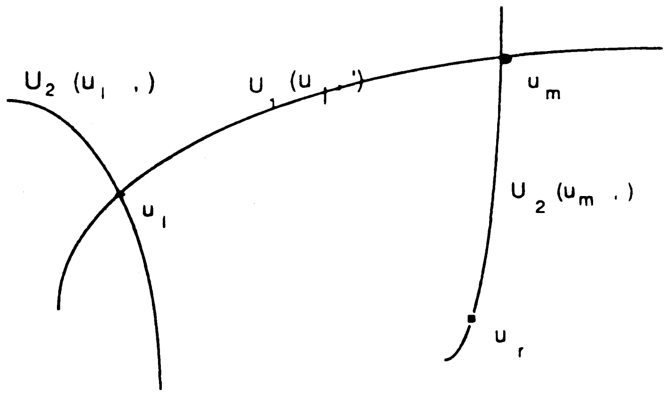


figure 3.



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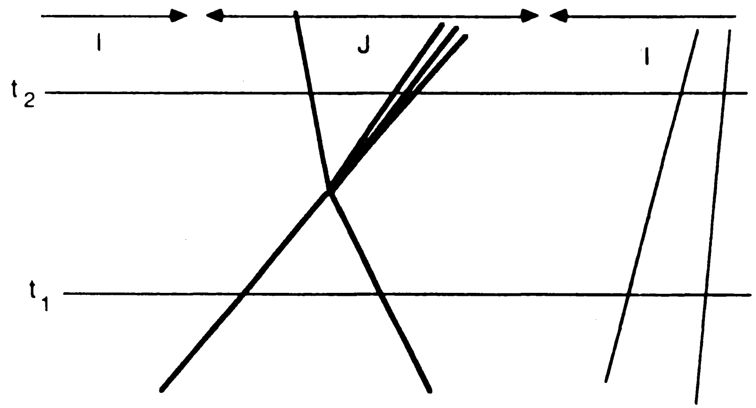

figure 4.